Steady State Solution of Continuum Model

Jinghao Chen
May 3, 2020

Steady state means the density doesn't change w.r.t time, i.e.

$$\frac{\partial \rho}{\partial t} = 0$$

2 results can be concluded from the equation above:

(1) $\rho(x,t) = \hat{\rho}(x)$ is only the function of spatial variable x, and as the composition, $K(\rho) = K\big(\rho(x,t)\big) = K\big(\hat{\rho}(x)\big)$ also only depends on x. We can redefine $\widehat{K}(x) = K\big(\hat{\rho}(x)\big) = K(\rho)$.

(2) Recall our continuum model $\frac{\partial \rho}{\partial t} = \frac{\partial^2 K(\rho)}{\partial x^2}$, so we have $\frac{\partial^2 K(\rho)}{\partial x^2} = 0$, which means $\frac{\partial^2 \widehat{K}}{\partial x^2} = 0$ implies $\widehat{K} = ax + R$, where a and R are constants.

From result (1), we have $\frac{\partial \rho}{\partial x} = \hat{\rho}'(x)$. So according to the homogeneous Neumann boundary conditions at $x = 0, 1$,

$$\frac{\partial \rho(x,t)}{\partial x}\Big|_{x=0} = \hat{\rho}'(0) = 0, \qquad \frac{\partial \rho(x,t)}{\partial x}\Big|_{x=1} = \hat{\rho}'(1) = 0$$

so

$$\frac{\partial K(\rho)}{\partial x}\Big|_{x=0} = K'(\rho) \cdot \frac{\partial \rho(x,t)}{\partial x}\Big|_{x=0} = 0$$
$$\frac{\partial K(\rho)}{\partial x}\Big|_{x=1} = K'(\rho) \cdot \frac{\partial \rho(x,t)}{\partial x}\Big|_{x=1} = 0$$

On the other hand

$$\frac{\partial K(\rho)}{\partial x}\Big|_{x=0,1} = \widehat{K}'\Big|_{x=0,1} = a = 0 \quad \Rightarrow \quad K(\rho) = \widehat{K} = R$$

Hence, we derive the following steady state equation

$$K(\rho) = \rho(1 - 2\alpha\rho + \alpha\rho^2) = R = const$$

Notes that we ρ represents the density, so we only consider the range of ρ from 0 to 1. Under that circumstance, according to

$$K'(\rho) = D(\rho) = 3\alpha\left(\rho - \frac{2}{3}\right)^2 + 1 - \frac{4}{3}\alpha$$

(1) Distributional solution has no jump $\Leftrightarrow$ graph of $K(\rho) = \rho(1 - 2\alpha\rho + \alpha\rho^2) = R$ is monotonic increasing (as α restricted on $[0,1]$ as well, so increasing is the only possible situation) $\Leftrightarrow$ $D(\rho) > 0$ $\Leftrightarrow$ $0 \le \alpha < \frac{3}{4}$ $\Leftrightarrow$ globally stable, PDE is well-posed

(2) Distributional solutions jump between ρ_1 and ρ_2 $\Leftrightarrow$ $K(\rho) = \rho(1 - 2\alpha\rho + \alpha\rho^2) = R$ has multiple roots for $\rho \in [0,1]$ $\Leftrightarrow$ $K(\rho) = \rho(1 - 2\alpha\rho + \alpha\rho^2)$ has 2 extrema for $\rho \in [0,1]$ $\Leftrightarrow$ $D(\rho) < 0$ $\Leftrightarrow$ $\frac{3}{4} < \alpha \le 1$ $\Leftrightarrow$ PDE is ill-posed in the interval I_α

In case (2), from the linear stability analysis, we need to get rid of that ρ located in the interval I_α where the diffusivity $D(\rho) < 0$, as we only allow for "stable" values of the density. From the shape of the curve $\rho(1 - 2\alpha\rho + \alpha\rho^2) = R$, it is easy to see that the dropped ρ is the middle one, the one between the minimal and maximal points. The remained 2 values ρ_1 and ρ_2 satisfy $K(\rho_1) = K(\rho_2) = R$, which match with the 2 plateau values in the results of numerical experiments.

Now we can generalize the weak solution (41) of steady state problem:

$$\rho = \begin{cases} \rho_1, & x \in [c_{2k}, c_{2k+1}) \\ \rho_2, & x \in [c_{2k-1}, c_{2k}) \end{cases}$$

And by the conservation of mass, we have

$$mass\ \ M = \rho_1 \cdot \sum_k (c_{2k+1} - c_{2k}) + \rho_2 \cdot \sum_k (c_{2k} - c_{2k-1})$$

As we consider $[0,1]$, $\sum_k (c_{2k+1} - c_{2k}) + \sum_k (c_{2k} - c_{2k-1}) = 1$, so define $C \coloneqq \sum_k (c_{2k+1} - c_{2k})$ and then $\sum_k (c_{2k} - c_{2k-1}) = 1 - C$. The mass equation becomes the following (43):

$$M = C\rho_1 + (1 - C)\rho_2$$

We see from this equality that mass M should be bounded below (or roughly speaking, big enough) to ensure the existence of jump solutions. To think about this in another way, we can construct ρ_1 and ρ_2 which satisfy equality (43) straddling M and satisfy $K(\rho_1) = K(\rho_2)$, then using (43) to determine C.