UCI Complex Analysis Qualifying Hints

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1st Edition

This booklet collects all Complex Qual problems from 2012 to 2019 with categorization, followed by hints or comments in few words. For the complete solutions, please refer to *UCI Complex Analysis Qualifying Solutions* in detail.

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Holomorphic Basic

S 2013 – 4, S 2018 – 4

Suppose $f(z) = u(x, y) + iv(y)$ is a holomorphic function. Show that there exists $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ such that $f(z) = az + \lambda$.

Hints:
Cauchy Riemann equation.

S 2015 – 1

Prove that for each $n \in \mathbb{N}$ every solution of the equation $(1 - iz)^n + z^n = 0$ must satisfy $\mathrm{Im}(z) = -\frac{1}{2}$.

Hints:
Directly plug in $z = x + iy$, middle school math required.

S 2016 – 5

5. Find all entire holomorphic functions f with justification such that

$$\mathrm{Im} f(z) = (y^2 - x^2),$$

where $\mathrm{Im} f$ denotes the imaginary part of f.

Hints:

Cauchy Riemann equation.

S 2016 – 6

6. Prove or disprove: there exists a family $\{f_n\}$ of holomorphic functions on $D(0,2)$ such that $f_n \to \bar{z}^3$ uniformly on the compact set $\{z \in \mathbf{C} : |z| = 1 \text{ or } 1/2\}$ (two circles: $|z| = 1$ and $|z| = 1/2$).

Hints:
Obvious result but not easy to prove it clearly. Try $z^2 f_n(z)$ to see what will happen on the integral around $\partial\mathbb{D}$. The same problem appeared again in Spring 2020 Qual.

S 2015 – 2

Classify all the singularities and find the associated residues for

$$f(z) = \frac{e^{-\frac{1}{z}}}{(z-1)(z+1)^2}$$

Hints:
The hardest part of this problem is the singularity at 0. It's tedious and cumbersome even after expansion. The trick here is to compute $Res_f(\infty)$.

F 2016 – 6

6. Prove or disprove there is a holomorphic function f on $\mathbf{C} \setminus D(0,3)$ such that

$$f'(z) = \frac{z^2+1}{z(z-1)(z-2)}$$

Hints:
[Residue at ∞, by Nick] Antiderivative exists if and only if $Res_f(\infty) = 0$, which can be computed by choosing big enough radius circle integral. Why that works? See the next problem. Remark:

$$\mathrm{Res}(f,\infty) = -\mathrm{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right) \qquad \mathrm{Res}(f,\infty) = \frac{-1}{2\pi i}\int_{C(0,r)} f(z)\,dz$$

[PBC, by Prof. Anton Gorodetski] the equivalent approach is to do a big circle integral, if antiderivative exist then by FTC the circle integral is 0. On the other hand, by Residue Theorem, that circle integral should be $2\pi i \sum Res_f$ where nonzero can derive contradiction.

S 2017 – 5

5. Let f be holomorphic in $D = \{z \in \mathbf{C} : 2 < |z| < \infty\}$ satisfying

$$\int_{|z|=3} f(z)dz = 0.$$

Prove that there is a holomorphic function F in D such that $F'(z) = f(z)$ on D.

Hints:
We see $Res_f(\infty) = 0$, so the Laurent expansion of ∞, all terms have antiderivatives (as 1/z disappeared), so exist.

F 2018 – 7

7. Let $f\colon \{z \,|\, |z| > 0\} \to \mathbb{C}$ be analytic. Furthermore suppose that $\lim_{z\to\infty} f(z) = 0$. Show that for $|z| > 1$, one has

$$\frac{1}{2\pi i}\int_{\eta=1} \frac{f(\eta)}{\eta - z} = -f(z).$$

Hints:
General Cauchy integral formula. The region is a big annulus contains fixed z, so the boundary where we need to let the integral go through with is 2 circles. The smaller one should be negative direction (clockwise) indicating outwards, while the bigger one should be positive direction (counterclockwise) as usual indicating inwards. Then we only need to show the bigger integral is zero, where we have 2 approaches: (1) Cauchy estimate, (2) residue at ∞, as $R \to \infty$ for both ways.

S 2019 – 4

4. Let

$$f(z) = \frac{z^3}{(z^2+1)\, e^{1/z}}$$

a)Find and classify all the singularities of $f(z)$ in the extended complex plane.
b) Evaluate

$$\oint_\Gamma f(z)dz,$$

where $\Gamma = \{z \in \mathbf{C} \mid |\mathrm{Re}\,\mathbf{z}| + |\mathrm{Im}\,\mathbf{z}| = \mathbf{3}\}$.
c) Does there is a holomorphic function F on $|z| > 3$ such that $F'(z) = f(z)$ on $|z| > 3$? (Justify your answer)

Hints:
Key is to correctly sketch the graph and use $Res_f(\infty)$ trick.

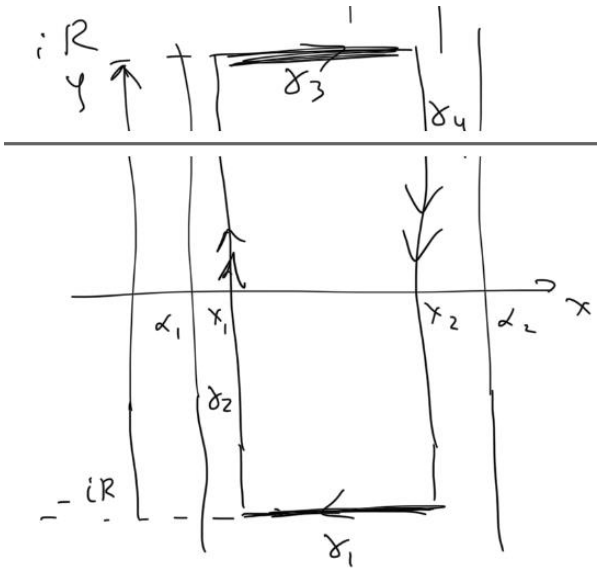
S 2006 – 4

4. Let $f(z)$ be analytic and satisfy $|f(z)| \leq 100|z|^{-2}$ in the strip $\alpha_1 \leq \mathrm{Re}\, z \leq \alpha_2$. Prove the function

$$h(x) = \int_{-\infty}^{\infty} f(x + i\, y)dy$$

is a constant function of $x \in [\alpha_1, \alpha_2]$.

Hints:



Drew by Prof. Gorodetski in 2020

Integral of $f(z)$ on γ_2 and γ_4 are $h(x_1)$ and $-h(x_2)$ respectively as $R \to \infty$, whereas on γ_3 and γ_1 are both 0 by Cauchy estimate through the given condition. The integral on overall rectangle is 0 by Cauchy integral theorem. Then we done.

S 2017 – 8

8. Prove or disprove there is a non-constant entire function $f = u + iv$ satisfying $v(z) \neq u(z)^2$ when $u(z) \geq 0$.

Hints:
W.L.O.G, $v \leq u^2$ on the right half plane of the range. So, the range of entire function f will miss a large part of unbounded region, which by common sense (well, I mean, by Picard theorem) should be const. To avoid Picard in the Qual (Picard will only get 7/10 if you tried), we just need to map the region of possible range conformally to $\mathbb{D}$, then by Liouville, we done.

F 2014 – 1

1. Let f be an entire holomorphic function such that $f(z) \notin \mathbf{R}$ for all $z \in \mathbf{C}$, where $\mathbf{R}$ is the real line in the complex plane $\mathbf{C}$. Prove or disprove f is a constant.

Hints:
Little Picard (just kidding. You may lose points by using this in the Qual)
Continuity → upper half plane to unit disc by Cayley → Liouville

F 2017 – 1

1. Let u be a real-valued continuous function on $\mathbf{C}$ such that $e^{u(z)}$ is harmonic in $\mathbf{C}$. Then u is a constant.

Hints:
Find the harmonic conjugate of $e^{u(z)}$ to form holomorphic $f(z)$, then entire function $f(z)$ will miss left half plane as the real function $e^{u(z)}$ always takes positive values, then by Liouville, we done.

F 2019 – 1

1. Complete the following parts.
(a) Find all points where the function $f(z) = \bar{z}^2 + 3\bar{z}$ is analytic

Hints:

Find the set of points where $\frac{\partial f}{\partial \bar{z}} \neq 0$

(b) Let D be a domain in $\mathbf{C}$ and let f be C^1 function on D such that

$$\int_{\partial D(z_0,r)} f(z)dz = 0,$$

for any $z_0 \in D$ and $0 < r < \delta(z_0) = \text{dist}(z_0, \partial D)$. Prove f is holomorphic in D.

Hints:
Morera theorem.

F 2019 – 5

5. Prove or disprove: there exist a sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$ on $D(0,1)$ such that $f_n(z) \to |z|^2$ uniformly on a non-empty open subset of $D(0,1)$.

Hints:
The trick that everyone who take Qual should know: $|z|^2 = z\bar{z}$

F 2012 – 3

3. Let $P(z)$ be a polynomial in z. Assume that $P(z) \neq 0$ for $\text{Re}(z) > 0$. Show that $P'(z) \neq 0$ for $\text{Re}(z) > 0$.

Hints:

A natural way to deal with polynomial is writing its product form w.r.t roots. Try $\frac{P'(z)}{P(z)}$.

PBC.

S 2018 – 2

Let f and g be analytic functions on the open set $U = D(1, 15) \setminus \{i\}$, i.e. the open disc centered at 1 with radius 15 with the point i removed. Suppose $f'(z) = g'(z)$ for all $z \in U$. Prove that f and g differ by a constant, that is, there exists $a \in \mathbb{C}$ such that $f(z) - g(z) = a$ for all $z \in U$.

Hints:
Taylor expansion + uniqueness principle (identity principle)

S 2018 – 7

Prove that the range of the function $f(z) = \sum_{n=1}^{2018} \cos^n(z)$ is the whole complex plane $\mathbb{C}$.

Hints:
Use definition proved the range of $\cos(z)$ is the whole complex plane, then redefine $\omega = \cos(z)$, so for any $\alpha \in \mathbb{C}$, $f(z) - \alpha$ is a polynomial w.r.t variable ω. As we already proved, ω can be any complex value as the range of $\cos(z)$ is the whole plane, and polynomial roots must exist on the plane, where ω must reachable. Similar problem refers to 2020 220C final.

S 2018 – 8

Let us say that a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ on the unit disc $\mathbb{D}$ is "good" if for some $n \in \{1, 2, \ldots, 2018\}$ the function f does not take values on the ray $\{te^{\frac{2\pi}{n}i} \mid t \geq 0\}$. Prove that collection of all "good" functions is normal.

Hints:
Continuity of holomorphic function → exist a subsequence all map to the same cut-region → equip conformal map by Riemann mapping theorem → uniformly

bounded, by Montel theorem

Holomorphic General and Singularities

F 2013 – 1

Describe all entire functions $f : \mathbb{C} \to \mathbb{C}$ which satisfy

a) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$ for all $n \in \mathbb{N}$;

b) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$ for all $n \in \mathbb{N}$.

Hints:
Identity principle.
This problem is also the problem 1 in the 2020 Gorodetski 220C midterm.

F 2017 – 2

2. Prove or disprove there is a holomorphic function f on the unit disk $D(0,1)$ such that

$$f(\frac{1}{2n}) = f(\frac{1}{2n+1}) = \frac{1}{n}$$

for all positive integers n.

Hints:
Subtraction, identity principle to find the general property, test by specific cases to get contradiction.

F 2018 – 3

3. Let $\{f_n\}$ be a uniformly bounded sequence of analytic functions on the open unit disc $D(0,1)$. Suppose $\lim_{n\to\infty} f_n\left(\frac{1}{k}\right)$ exists for $k = 1, 2, \ldots$. Prove that there exists an analytic function f on $D(0,1)$ such that $f_n \to f$ uniformly on compact subsets of $D(0,1)$.

Hints:
Montel theorem says there exists a convergent subsequence, which has a holomorphic limit f. Then PBC (proved by contradiction) says f_n has a limit function $\tilde{f}$. Finally say $f = \tilde{f}$ as they agree on an accumulative set.

F 2013 – 2

Suppose f is an entire function such that $\lim_{z\to\infty} \frac{|f(z)|}{|z|} = 0$. Prove that f is constant.

S 2014 – 1 (b)

(b) Describe all entire holomorphic functions f with $\lim_{z\to\infty} \frac{f(z)}{z} = 0$.

Hints:
Sublinear growth, Theorem 3.4.4
Mainly it's Cauchy integral formula

S 2016 – 4

4. Let f be an entire function. Prove the following two statements.
(a) If $|f(z)| \leq M(1 + |z|^n)$ on $\mathbf{C}$ for some positive constant M then f is a polynomial of degree at most n.

(b) If $\lim_{|z|\to\infty} |f(z)| = \infty$ then f is a polynomial.

Hints:
Classical one. First part, an important way to prove polynomial is to show that after differentiating finitely many times the function would become 0. Use Cauchy estimate to do it. The second part, use $f\left(\frac{1}{z}\right)$ to see 0 is a pole, then upside down back.

F 2013 – 7

Assume that $z = 0$ is an essential singularity of a holomorphic function f. Show that it is also an essential singularity of f^2, f^3, and, in fact, of f^n for every $n \in \mathbb{N}$.

Hints:
Casorati-Weierstrass Theorem, the weaker Great Picard.

S 2014 - 1

1. Complete the following two problems.
   (a) Describe all entire holomorphic functions f with $|f(z)| \leq |z|$ for all $z \in \mathbf{C}$;

Hints:

Method 1: take $g := \frac{f}{z}$ → Riemann removable singularity theorem → Liouville

Method 2: Theorem 3.4.4, sublinear growth case

F 2016 – 3

3. Prove or disprove there is a non-zero holomorphic function f in the complex plane $\mathbf{C}$ such that

$$|f(z)|^2 \leq |\cos z|, \quad z \in \mathbf{C}.$$

Hints:

Check $g := \frac{f^2}{\cos z}$, by Riemann removable singularity theorem, g is actually an entire function. Then by Liouville.

S 2017 - 7

7. Let f be meromorphic in $\mathbf{C}$ satisfying

$$|f(z)|^3 \leq |\tan z|, \quad z \in \mathbf{C} \setminus P(f),$$

where $P(f)$ is the set of poles of f in $\mathbf{C}$. Prove $f(z) \equiv 0$.

Hints:

Put together, Riemann removable singularity theorem + Liouville theorem

Use the multiplicity of zero to derive contradiction in the end.

F 2018 – 2

2. Let f and g be entire functions. Suppose that

   (a) $g(z) \neq 0$ for all $z \in \mathbb{C}$,

   (b) $|f(z)| \leq \left|z^7 g(z)\right|$ for all $z \in \mathbb{C}$.

   Prove that there exists $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ such that $f(z) = \alpha z^7 g(z)$ for all $z \in \mathbb{C}$.

Hints:
Put together, Riemann removable singularity theorem + Liouville theorem

S 2019 – 1

1. Let $f(z)$ be analytic in the region $0 < |z| < 1$, which satisfies $\mathrm{Re} f(z) < 2$. Show that $z = 0$ is a removable singularity of f.

Hints:
Create e^f which is still holomorphic, then by Liouville.

F 2016 – 4

4. Let f be meromorphic in the complex plane $\mathbf{C}$ such that

$$|f(z)| = 1 \quad \text{on} \quad |z| = 1.$$

Prove f is a rational function.

Hints:
We know that in $\mathbb{D}$, the numbers of zeros and singularities are finite (as they have to be isolated), so we can divide $f(z)$ by Mobius transformation of those points, where we just drop those points out of unit disc. Then, by Maximum Modulus Principle, new function after dropping should be const. We done.

F 2019 – 6

6. Suppose that f is a non-constant entire function which satisfying

$$|f(z)| \geq 1 \quad \text{when} \quad |z| \geq 10.$$

Prove that f is a polynomial.

Hints:
Casorati-Weierstrass, ∞ should be removable or pole for the function.
Discuss about these 2 kinds separately.

F 2012 – 6

6. Let $f(z)$ be an entire holomorphic function on $\mathbf{C}$ such that

$$|f(e^z)| \leq |e^z|, \quad z \in \mathbf{C}$$

Prove $f(z) \equiv az$ for some constant $|a| \leq 1$.

Hints:

The range of e^z is $\mathbb{C}\backslash\{0\}$, consider $g \coloneqq \frac{f}{z}$, by inequality we have 0 is removable for g. Hence g is entire, by Liouville is const.

S 2014 – 8

8. Let $f(z)$ be holomorphic in $D =: D(0,1) \setminus \{0\}$ such that

$$\int_D |f(z)| dA(z) < \infty$$

Prove that $z = 0$ is either removable or a simple pole .

Hints:
Relatively hard problem.
If the integrant is $|f(z)|^2$, then we can use Laurent expansion to cancel something. But here is not. So use MVP, showing that it is removable， simple pole or pole of order 2, finally disprove pole of order 2.

F 2012 – 7

7. Let f be holomorphic in $D(0,1) \setminus \{0\} = \{z \in \mathbf{C} : 0 < |z| < 1\}$. If

$$\int_D |f(z)|^3 dA(z) = \int_D |f(x+iy)|^3 dxdy < \infty$$

then $z = 0$ is removable singularity of f.

Hints:
MVP on $D(z, |z|)$. Directly get it.

F 2017 – 7

7. Let $\mathcal{F}$ be a family of holomorphic functions f on the unit disc $D(0,1)$ such that

$$|f(0)|^2 + \int_{D(0,1)} |f'(z)|^2 dA \leq 1.$$

Prove that $\mathcal{F}$ is a normal family.

Hints:
FTC, enlarge through Holder inequality to see uniformly bounded, then by Montel theorem, we done.

F 2014 – 5

5. Let D be a bounded domain in $\mathbf{C}$ with piecewise C^1 boundary. Let $f(z)$ be holomorphic in a bounded domain D and $f \in C(\overline{D})$ with all zeros $\{z_1, \cdots, z_n\} \subset D$ counting multiplicity. Let g be holomorphic in D and continuous on $\overline{D}$. Evaluate

$$\int_{\partial D} \frac{f'(z)}{f(z)} g(z) dz$$

Hints:
Naturally way of thinking, take zeros of f out, then simplify. $i.e.$ $f = \prod(z - z_k)h$ and then plug in to see what happens.

F 2017 – 4

4. Let $z_1, \cdots, z_n \in D(0, R)$ and

$$Q(z) = (z - z_1) \cdots (z - z_n)$$

Let f be a holomorphic function on $\overline{D(0, R)}$. Prove

$$P(z) = \frac{1}{2\pi i} \int_{|w|=R} f(w) \frac{Q(w) - Q(z)}{(w - z)Q(w)} dw$$

is a polynomial of degree $n - 1$ such that $f(z_j) = P(z_j)$ for $1 \leq j \leq n$.

Hints:
Split according to the numerator, one by Cauchy integral formula, one by residue theorem. During the process, just fixed z as a point not equal to z_j for any j. After that, try $z = z_j$ to derive the second conclusion.

S 2015 – 6

Let the function $f(z)$ be meromorphic in a neigbourhood of the unit disk $\{|z| \leq 1\}$ and suppose it has only one singular point z_0 on the circle $|z| = 1$ which is a simple pole. Show that $\frac{f^{(n)}(0)}{n!} = \frac{A}{z_0^n}(1 + \phi_n)$ where $\lim_{n\to\infty} \phi_n = 0$.

Hints:
There are 2 ways to derive this simple pole:
1. Direct extract，i.e. $f = (z - z_0)h$.
2. Laurent expansion

Latter one is better in this problem.

F 2017 – 3

3. Let f be an entire holomorphic function in $\mathbf{C}$ such that $f(x)$ and $f(ix)$ are real for $x \in (1, 2)$. Prove there is an entire function g such that

$$f(z) = g(z^2), \quad z \in \mathbf{C}.$$

Hints:
Key idea, is to show the expansion of $f(z)$ has no odd power terms. Test by real $x \in (1,2)$ first to show all coefficients of the Taylor expansion are real, then test ix in the expansion to show the odd power terms should be 0. This process should use identity principle by constructing subtraction functions, which is not straightforward. (as you can't test by few values and say the properties hold for all points)

F 2018 – 1

1. Let f be an analytic function on $D(z_0, r) \setminus \{z_0\}$, where $r > 0$, such that $f(z) \neq 0$ for all $z \in D(z_0, r) \setminus \{z_0\}$. Consider the analytic function $g(z) = \frac{1}{f(z)}$ for $z \in D(z_0, r) \setminus \{z_0\}$. Prove that f has an essential singularity at z_0 if and only if g has an essential singularity at z_0.

Hints:
Casorati-Weierstrass

F 2019 – 7

7. Let $f : D(0,1) \to D(0,1)$ be a proper holomorphic map such that $f(z)$ is continuous on $\overline{D(0,1)}$. Prove f is a rational function.

Hints:
Map boundary to boundary. Divided by Mobius transforms of zeros to make nonzero holomorphic g, then apply maximum modulus theorem on both g and $\frac{1}{g}$ to argue g is const.

S 2018 – 5

Suppose $p(z)$ is a polynomial of degree $d \geq 2$ that has only simple zeros $r_1, r_2, \ldots, r_d$. Prove that $\frac{1}{p'(r_1)} + \frac{1}{p'(r_2)} + \ldots + \frac{1}{p'(r_d)} = 0$.

Hints:
$Res_{\frac{1}{p(z)}}(r_j) = \frac{1}{p'(r_j)}$, so the given LHS is the sum of all residues of simple poles of $\frac{1}{p(z)}$.
This value equals $-Res_{\frac{1}{p(z)}}(\infty)$, which is equal to 0 as ∞ is removable for $\frac{1}{p(z)}$.

Power Series

S 2013 - 1

Find the largest set D where the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} z^{n^2}$$

converges.

Hints:
Abel, ratio test

S 2014 - 4

4. Prove that there is an N such that if $n \geq N$ then

$$\sum_{k=0}^{n} (k+1) z^k \neq 0, \quad z \in D(0, 3/4)$$

Hints:
Obviously, this guy is the result of the derivative of a series, so integrate back, sum up, find the limit, and finally use Hurwitz theorem PBC.

F 2014 – 4

4. Let $h(x)$ be a twice differentiable function on $[-1,1]$ such that $h(0) = h'(0) = 0$ and $h''(0) \neq 0$. Prove

$$\sum_{n=1}^{\infty} h(\frac{1}{n}) z^n$$

defines a holomorphic function on $D(0,1)$ which is continuous on $\overline{D(0,1)}$.

Hints:
Abel criterion to show the radius of convergence, we need to do Taylor expansion for real function h.

S 2015 – 3

Expand in a series of powers of w each of the branches of $z\,(w)$ defined by the equation $w = 2z + z^2$ (for one branch $z\,(0) = 0$, for the other $z\,(0) = -2$).

Hints:
Solve z(w), then expansion as usual.

F 2015 - 1

Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly in the unit disc $|z| \leq 1$. Does the series obtained by term-by-term differentiation converge uniformly in the unit disc? Explain your answer.

Hints:
Not.

F 2015 – 2

Let f be an entire function and suppose that there exists a bounded sequence $\{a_n\}$ of real numbers such that $f(a_n)$ is real for all $n \in \mathbb{N}$. Prove that $f(x)$ is real for all real x.

Hints:
Method 1 (expansion way): $u(x) = Re(f)|_{\mathbb{R}}$, Taylor expansion around 0, then replace all x in the expansion by z, that expansion converges on $\mathbb{C}$ by Abel criterion, so defines an entire function g. g and f agree on an accumulation set (from the condition by Bolzano-Weierstrass), so by identity principle, $f = g$.

Method 2 (Schwarz type construction).

F 2015 – 7

TRUE or FALSE: The family $\mathcal{F}$ of functions holomorphic in a unit disc with power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that satisfy $|a_n| \leq n^{2015}$ is normal.

Hints:
Montel theorem, version 2.

S 2016 – 1

1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$$

defines a meromorphic function on $\mathbf{C}$.

Hints:

Just do it by definition.

S 2016 – 3

3. For any non-integers α, β and γ, find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)} z^n.$$

Hints:
Ratio test. That's it.

F 2016 – 5

5. Find the radius of convergence for

$$\sin\left(\frac{2}{(z-2i+2)(z-3+i)}\right) = \sum_{n=0}^{\infty} a_n z^n$$

Hints:
Radius should be the shortest distance between expansion center (here is 0) and singularities.

S 2017 – 2

2. The Bernoulli polynomials $B_n(z)$ are defined by the expansion

$$t\frac{e^{tz}-1}{e^t-1}=\sum_{n=1}^{\infty}\frac{B_n(z)}{n!}t^n.$$

Prove that $B_n(z+1)-B_n(z)=nz^{n-1}$.

Hints:
Do subtraction of the whole series.

S 2019 – 5

5. Assume that the power series

$$f(z)=\sum_{j=0}^{\infty}a_jz^j$$

defines a holomorphic function for $|z|<1$ and assume that

$$|a_1|>\sum_{j=2}^{\infty}j\,|a_j|\,.$$

Show that the function $f(z)$ is one-to-one in the open unit disk $D(0,1)$.

Hints:
It suffices to show injective by proving $f'(z)\neq 0$. Just directly use inequality.

S 2019 – 6

6. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ denote an entire function satisfying the estimate

$$|f(z)| \leq 10e^{|z|}, \quad \text{for all } z \in \mathbf{C}.$$

Prove that the coefficients a_j satisfy

$$|a_j| \leq 10\big(e/j\big)^j, \quad \text{for all } j \in \mathbf{N}.$$

Hints:
Use integral expression for the coefficients of the expansion, then use Cauchy estimate, and finally replace radius by j as function is entire so arbitrary radius holds.

F 2019 – 3

3. Given a series

$$\sum_{n=1}^{\infty} \left(\frac{2019+i}{2019-i}\right)^{n^2} \cdot \left(\frac{z-2019}{z+2019}\right)^n.$$

(i) Find all complex numbers z such that the series converges absolutely;
(ii) Find all complex numbers z such that the series convergence.

Hints:
Key: redefine first part by a_n, redefine second part by ω^n, then we turn that guy into a regular Taylor series.

F 2019 – 8

8. (a) Prove

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2} - \frac{\pi^2}{\sin^2(\pi z)}$$

is entire holomorphic.

Hints:
Prove all those singularities are removable.

(b) Prove

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2} = \frac{\pi^2}{\sin^2(\pi z)}, \quad z \notin \mathbf{Z}.$$

Hints:
This problem is not easy at all.

Example. We show that $\pi^2/\sin^2(\pi z)$ has the partial fractions decomposition

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}. \tag{2.1}$$

Set $f(z) = \sum 1/(z-k)^2$. If $|z| \le R$, and if $|k| > 2R$, then $|z-k| > |k|/2$, and $1/|z-k|^2 < R/k^2$. By the Weierstrass M-test, the series converges uniformly for $|z| \le R$. Thus $f(z)$ is a meromorphic function on the complex plane. The functions $f(z)$ and $\pi^2/\sin^2(\pi z)$ have the same poles and principal parts, so that $f(z) - \pi^2/\sin^2(\pi z)$ is entire. Evidently, $f(z+1) = f(z)$, so that $f(z) - \pi^2/\sin^2(\pi z)$ is periodic with period 1. We may focus on the period strip consisting of $z = x + iy$ for which $0 \le x \le 1$. In this strip we have

$$\left|\frac{1}{(z-k)^2}\right| = \frac{1}{y^2 + (k-x)^2} \le \frac{1}{(k-1)^2}, \qquad |k| \ge 2, \ |y| \ge 1.$$

By the Weierstrass M-test, the series for $f(z)$ converges uniformly and $f(z)$ is bounded for $|y| \ge 1$ and $0 \le x \le 1$. Since each summand tends to 0 as $|y| \to \infty$, the uniform convergence implies that $f(x+iy) \to 0$ as $|y| \to \infty$, $0 \le x \le 1$. The identity $|\sin z|^2 = |\sin x|^2 + |\sinh y|^2$ shows that also $\pi^2/\sin^2(\pi z)$ is bounded for $|y| \ge 1$ and tends to 0 as $|y| \to \infty$. Thus $f(z) - \pi^2/\sin^2(\pi z)$ is bounded for $|y| \ge 1$, hence bounded on the vertical strip $\{0 \le x \le 1\}$, hence by periodicity bounded on the entire complex plane. By Liouville's theorem it is constant. Since both $f(z)$ and $\pi^2/\sin^2(\pi z)$ tend to 0 as $|y| \to \infty$, the constant is zero, and $f(z)$ coincides with $\pi^2/\sin^2(\pi z)$.

F 2012 – 8

8. Find the largest set in $\mathbf{C}$ where the Laurent series

$$\sum_{j=-\infty}^{\infty} 2^j z^{j^3}$$

converges.

Hints:
Laurent series converges if and only if both principal part and Taylor expansion part convergent. This guy is actually divergent everywhere.

Integral Calculation

S 2013 - 7

Evaluate the integral for $a > 0$

$$\int_{-\infty}^{+\infty} \frac{\cos^3 x}{a^2 + x^2} dx$$

Hints:

Derive $\cos^3 z = \frac{1}{4}(\cos 3z + 3\cos z)$ from $\cos z = \frac{e^{iz}+e^{-iz}}{2}$

Then Residue theorem + Cauchy estimate

F 2012 – 1

1. Show that for $a > 0$,

$$\int_0^\infty \frac{\cos ax}{(1+x^2)^2} dx = \frac{\pi(1+a)}{4e^a}.$$

Hints:
Keywords: even function, real part of exponential function, choose proper exponential function or proper region to make Cauchy estimate work.

F 2013 – 4

Evaluate the integral

$$\int_0^\infty \frac{1}{1+x^5} dx$$

Hints:

1. Sector cutting, the angle is not arbitrary.
2. $\frac{P}{Q}$ type to compute simple pole residue is faster than real number L'Hopital。
3. Linear fractional part $\sim O(\frac{1}{R^5})$, arc of sector using Cauchy estimate。
4. Cauchy estimate, careful about dealing with the denominator, as the incorrect manipulation can be result in correct conclusion.
5. General case $x^5 \to x^n$ can be a part of other Qual problem（e.g. F 2014 - 2）， or be a problem itself（e.g. next one S 2016 - 2）。

s

S 2016 – 2

2. Show that for a positive integer $n \geq 1$

$$\int_0^\infty \frac{1}{x^{2n}+1}\,\mathrm{dx} = \frac{\pi}{2\mathrm{n}\sin\frac{\pi}{2\mathrm{n}}}.$$

Hints:
General case

.

S 2014 - 2

2. Complete the following two problems.
   (a) Evaluate $\int_{|z|=1} \exp(\frac{1}{z^2})dz$. (Here, $\exp(z) =: e^z$)

Hints:
Method 1: Laurent expansion of exp → take a close look
Method 2: Residue theorem
Both of them are related to a_{-1}

S 2014 - 2

(b) Evaluate $\int_0^\infty \frac{x^2}{1+x^4}dx$

Hints:
Even function, use the whole real line to do integral, and cut in the end.

F 2014 – 2

2. Evaluate the real integral

$$\int_0^\infty \frac{\ln x}{1+x^4} dx$$

Hints:
Tedious calculation.
Know $\ln z = \ln|z| + i\,\mathrm{Arg}(z)$.

F 2017 – 6, F 2019 – 4

6. Prove

$$\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$$

Hints:
A super tedious problem. Need to work out on it before Qual.
Shape is a half annulus. For the small half circle, 2 ways: (1) Cauchy estimate just like big half circle, (2) residue theorem on 0, but half. Singularity 0 is a simple pole, so just use limit to compute its residue. For the left straight line, expected to be separated into 3 parts as it has a square on the top. One is our target integral, one we can find antiderivative easily, the only left we also have 2 ways: (1) [Prof. Zhiqin Lu] split the integral interval by 1, then 2 parts are actually symmetric with opposite signs, so we can cancel them. This method is a little bit tricky in the technique way. (2) [Prof. Anton Gorodetski] just keep its form until the last step. We only care about the real part (as we are doing the real integral), and this guy is in the imaginary part, so we just get rid of it. This method is also tricky more naturally in the complex analysis.

S 2015 – 4

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx$$

Hints:

1，$\sin \pi z = Im\left(e^{i\pi z}\right)$, nearly unsolvable if use definition of $\sin(z)$.

2，Linear fractional part $\sim O\left(\frac{1}{R}\right)$，Cauchy estimate doesn't work, need to use Fei estimate。

S 2018 – 6

Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x+i} dx$

Hints:

Again, Fei estimate.

F 2015 - 3

Evaluate

$$\int_0^{\infty} \frac{\sin ax}{x(x^2 + b^2)} dx, \quad a, b > 0.$$

Hints:

Again, even function trick, integrating the whole real line.

Here we have a little trick: to avoid the singularity at 0, a half circle integral is required. Its value is exactly the half of the integral of the whole circle, so we can use residue theorem. Notes that only half circle works. For 1/3, 1/4 circles, analogue results are not

guarantee.

F 2016 – 1

1. Evaluate the following integral

$$\int_0^\infty \frac{x}{(1+x^5)} dx.$$

Hints:
Sector cutting.

S 2017 – 1

1. Find the integral

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}, \quad a > 1.$$

Hints:
Use circle integral, rewriting $\cos(\theta)$ into the form with e^z.

F 2018 – 5

5. Evaluate $\int_\gamma \frac{1+z}{1-\cos(z)} dz$, where
   (a) γ is the circle of radius 5 around 0, counterclockwise.
   (b) γ is the circle of radius 7 around 0, counterclockwise.

Hints:

Essential singularities. Use expansion, then determine the coefficients.

Conformal Mapping

S 2013 – 8

Find explicitly a conformal mapping of the domain

$$U = \{|z| < 1, z \notin [1/2, 1)\} = \mathbb{D}\backslash[1/2, 1)$$

to the unit disc $\mathbb{D} = \{|z| < 1\}$.

Hints:
The key here is to create an "empty" radius, then use $\sqrt{z}$ to make it be a half disc. Use Mobius transformation to turn $[1/2,1]$ into $[0,1]$.

F 2013 - 3

Describe explicitly the automorphism group $Aut(\mathbb{C}\backslash\{0, 1\})$.

Hints:
Consider the permutation of 0, 1 and ∞ those three points.

S 2014 - 6

6. Let $D = \{z \in \mathbf{C} : 1 < |z+1| \text{ and } |z+2| < 2\}$. Construct a conformal holomorphic map which maps D onto the unit disc $D(0,1)$

Hints:

Use $\frac{1}{z}$ to map into a strip, then move it to the strip between 0 and πi, and use exp maps to $\mathbb{H}$ upper half plane, then Cayley.

F 2014 -6

6. Let $D = \{z \in \mathbf{C} : |z| < 1, \Re z > 0, \Im z > 0\}$. Construct a conformal holomorphic map which maps D onto the unit disc $D(0,1)$

Hints:

Too easy to be a Qual question.

F 2015 – 8

Set $U_1 = \{1 < |z| < 2\}$ and $U_2 = \{0 < |z| < 1\}$.

a) Show that homeomorphism $f : U_1 \to U_2$ given by $f(re^{i\theta}) = (r-1)e^{i\theta}$ is not a conformal mapping;

b) Does there exist a conformal mapping $g : U_1 \to U_2$?

Hints:

Although bijective, it's not even holomorphic. Check the form in terms of z.

The second part, PBC. Conformal map will map boundary point to boundary point, and

map interior point to interior point.

S 2016 – 7

7. Construct a conformal map ϕ which maps D_1 onto D_2, where

$$D_1 = \{z = x + iy \in D(0,1) : y > x\}; \text{ and } D_2 = \{z \in \mathbf{C} : |z| > 1\}.$$

Hints:
Too easy.

S 2017 – 6

6. Find a conformal map which maps U_1 onto U_2, where

$$U_1 = \{z = x+iy \in \mathbf{C} : y > 0\} \backslash \{z = iy : 1 \leq y \leq 2\} \quad \text{and} \quad U_2 = D(0,1) \backslash \{0\}.$$

Hints:
Think about F 2015 - 8, it seems this cannot be found if we see "conformal" concept as bijective.

S 2018 – 3

Let $L \subset \mathbb{C}$ be the ray $\{t + it \mid t \geq 1\}$, and $U = \{\operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}$. Find an explicit conformal mapping of $U \backslash L$ to the unit disc.

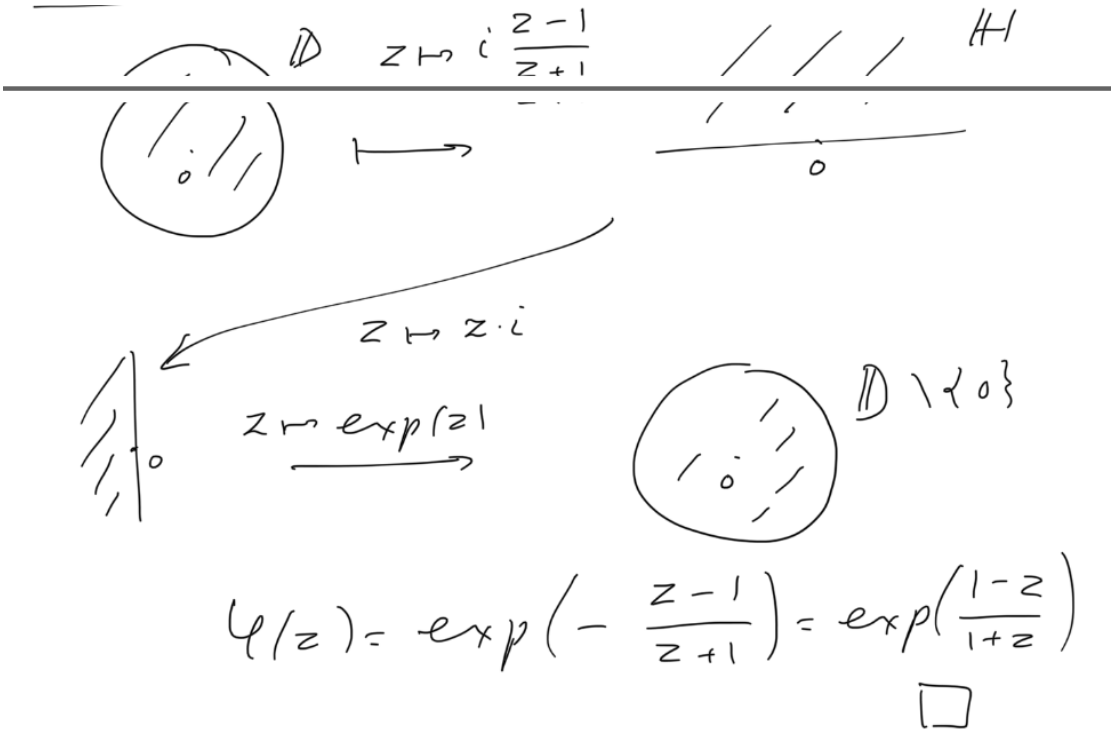
Hints:
Easy. Try $z \to z^2$ first and then Cayley. You'll see.

F 2018 – 6

6. Find a surjective holomorphic map φ from the open unit disc $D = D(0,1)$ to the punctured disc $D^* = D \setminus \{0\}$, with $\varphi'(z) \neq 0$ for any $z \in D$.

Hints:
This does not mean conformal map. The simply connected $\mathbb{D}$ is not conformally equivalent to punctured $\mathbb{D}\backslash\{0\}$. We just need to construct several conformal maps and on the final step construct a locally bijective map.

[Prof. Anton Gorodetski]



S 2019 – 7

7. Construct a conformal map ϕ which maps D_1 onto D_2, where

$$D_1 = \{z \in \mathbf{C} : 0 < \mathrm{Re} z < 2\} \text{ and } D_2 = \{z \in \mathbf{C} : |z| > 1\}.$$

Hints:
Turn the banded region into upper half plane, then Cayley.

Counting Roots

F 2012 – 4, S 2013 – 5

Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

inside the annulus $1 \leq |z| \leq 2$.

Hints:
Rouche on both circles, the outer circle should be set a little bit bigger radius

S 2018 – 1

Determine the number of roots, counted with multiplicity, of the equation

$$2z^5 - 15z^2 + z + 2$$

inside the annulus $1 \leq |z| \leq 2$.

Hints:
Rouche

F 2013 – 6

How many solutions has the equation

$$z^4 + 3z^2 + z + 1 = 0$$

in the closed upper half unit disc?

Hints:
Hardest Rouche problem.
Real coefficient polynomials have conjugate complex roots, so we consider the unit disc first. Check both boundaries then (straight line and arc).

F 2015 – 4

How many roots does the equation $z + e^{-z} = a$, $a \in \mathbb{R}, a > 1$, have in the right half plane?

Hints:
Create a half disc on the right half plane, and then consider the limit when radius of this disc goes to infinity so that it fills the whole half plane.
Keep calm and use Rouche.

F 2017 – 5

5. For $a > 1$ Prove the equation $ze^{a-z} = 1$ has a unique solution in $|z| \leq 1$, which is also real and positive.

Hints:
Try $\partial\mathbb{D}$ first to show no root on it, then use Rouche theorem to test interior.

F 2018 – 4

4. Find the number of solutions with multiplicity of $e^z = 7z^9$ in the open unit disc around the origin.

Hints:
Typical Rouche

F 2015 – 6

Consider a non-constant polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots a_1 z + a_0, \quad n \geq 1, \quad a_0, a_n \neq 0,$$

and set $B = \max_{0 \leq j \leq n-1} |a_j|$, $C = \max_{1 \leq j \leq n} |a_j|$. Prove that all roots of the polynomial P lie inside the annulus $r \leq |z| \leq R$, where

$$r = \frac{1}{1 + \frac{C}{|a_0|}}, \quad R = 1 + \frac{B}{|a_n|}.$$

Hints:
Again, keep calm and summon Rouche.

F 2019 – 2

2. Let $f : D(0,1) \to D(0,1)$ be holomorphic such that $f(0) = 5^{-20}$. Give a sharp estimate for number of zeros of f on $\overline{D(0, \frac{1}{5})}$.

Hints:
The strangest root-counting problem in the qual I've ever met.
Use Jensen's formula.

Holomorphic Functions on Geometric Domains

S 2013 - 3

Suppose f is holomorphic on the upper half plane $\mathbb{H} = \{\text{Im}\, z > 0\}$, $f(i) = 0$, and $|f(z)| \leq 1$ for all $z \in \mathbb{H}$. Prove that $|f(2i)| \leq \frac{1}{3}$.

Hints:
Schwarz-Pick or Schwarz Lemma
$\mathbb{H}$ is the same as $\mathbb{D}$.

F 2013 – 8

Suppose f is holomorphic on $\mathbb{C}\backslash\{0\}$, and suppose that $f\left(\frac{1}{z}\right) = f(z)$ for all $z \in \mathbb{C}\backslash\{0\}$. Suppose further that f is real on the unit circle. Show that f is real for all real $z \neq 0$.

Hints:
Method 1: Schwarz reflective principle on the circle (or + conformal map)
Method 2: little tricks about unit circle and conjugate + identity principle

F 2012 – 5

5. Let f be analytic on the upper-half plane and satisfy $|f(z)| < 1$. Furthermore suppose $f(i) = 0$. Give an upper bound for $|f'(i)|$ and state which functions realize this extrema.

S 2014 - 3

3. Let $f : D(0,1) \to D(0,1)$ be holomorphic with $f(0) = \frac{1}{3}$.
(a) Give a sharp upper bound estimate for $|f'(0)|$

(b) Give an example of f such that $|f'(0)|$ achieves the upper bound you obtained in Part (a)

S 2015 – 5, F 2018 - 8

Suppose $f : D(0,1) \to D(0,1)$ is a holomorphic mapping and $f(0) = \frac{1}{5}$. Give an upper bound for $|f'(0)|$, and characterize the functions for which the upper bound is an equality.

S 2016 - 8

8. Let $f : U \to U$ be holomorphic with U being the upper half plane. Prove that

$$|f'(i)| \leq |f(i)|$$

and provide an example indicates the above inequality is an equality.

F 2016 – 7

7. Let f be analytic on the upper-half plane and satisfy $|f(z)| < 1$. Furthermore suppose $f(2+i) = 0$. Give an upper bound for $|f'(2+i)|$ and state which functions realize this extrema.

S 2019 – 3

3. a) State the Schwarz-Pick Lemma.

b) Suppose $f : D(0,1) \to D(0,1)$ is a holomorphic mapping such that $f(0) = \frac{1}{6}$. Give an upper bound for $|f'(0)|$, and characterize the functions for which the upper bound is an equality.

Hints:
Schwarz-Pick or Schwarz lemma

S 2014 – 7

7. Let D be a simply connected domain in $\mathbf{C}$ and $z_0 \in D$. If $\phi_1, \phi_2 \in \mathrm{Aut}(D)$ such that

$$\phi_1(z_0) = \phi_2(z_0) \quad \text{and} \quad \phi_1'(z_0) = \phi_2'(z_0)$$

then $\phi_1 \equiv \phi_2$. (Hint: Try $D = D(0,1)$ and $D = \mathbf{C}$ first)

Hints:
Simple connected domains in $\mathbb{C}$ are conformally equivalent to $\mathbb{D}$ or $\mathbb{C}$。Discuss them separately.

F 2014 – 3

3. Let f be entire holomorphic such that $f(x+ix) \in \mathbf{R}$ for all $x \in \mathbf{R}$. If $f(2) = 1 - i$ then find $f(2i)$, where $i^2 = -1$.

Hints:
Conformal map + Schwarz reflexive principle。

F 2014 – 7

7. Let D be a simply connected domain in $\mathbf{C}$ and $z_0 \in D$. Let $\mathcal{F}$ be the set of all $f : D \to D(0,1)$ such that (i) $f(z_0) = 0$; (ii) $f'(z_0) > 0$ and (iii) f is one to one. Then prove $\mathcal{F}$ is not empty set.

Hints:
Only consider $\mathbb{D}$ or $\mathbb{C}$ by Riemann Mapping theorem.
Consider different automorphism types to determine the coefficients.

S 2017 – 3

3. Let $f(z)$ be analytic in $S = \{z = x + iy : -1 < x < 1\}$ and continuous on $\overline{S}$, the closure of S. Suppose that $f(z)$ are real when Re $z = x = \pm 1$. Prove that $f(z)$ can be extended analytically to the whole plane and that the resulting entire function satisfies $f(z + 4) = f(z)$ for all $z \in \mathbf{C}$.

Hints:
Schwarz reflexive principle
Get the idea is easy, manipulation is tedious. Think about 1~3 first, then 3~5 from 1~3, and then find out this guy is exactly periodic w.r.t -1~1.

Harmonic and Subharmonic Functions

S 2013 - 2

Let $\{f_j\}$ be a sequence of holomorphic functions from $D(0,1)$ to $D(0,1) \setminus \{0\}$. Prove that if $\sum_{j=1}^{\infty} |f_j(0)|$ converges, then $\sum_{j=1}^{\infty} f_j(z)^2$ converges absolutely and uniformly on compact sets in $D(0,1/3)$.

Hints:
Harnack principle, or Harnack inequality
The absolute value of a holomorphic function without zeros is subharmonic ($e^{\ln|f|}$ can be defined), but not harmonic. $\ln|f|$ is harmonic.

S 2014 – 9

9. Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be a sequence of holomorphic functions with $\sum_{n=1}^{\infty} |f_n(0)|^2 < \infty$. Prove that

$$\sum_{n=1}^{\infty} |f_n(z)|^3$$

converges uniformly on $\overline{D(0,1/5)}$.

Hint:
MVP（connect subharmonic with harmonic） + Harnack

F 2013 – 5

Prove that the function

$$u(x,y) = y\cos y \sinh x + x \sin y \cosh x$$

is harmonic in $\mathbb{R}^2$ and find its harmonic conjugate.

Hint:
Solve ODE by definition.

S 2014 – 5

5. Let $f_1, \cdots, f_n$ be holomorphic in a domain D in $\mathbf{C}$ and $p \in (0, \infty)$. Prove
   (a) $\sum_{j=1}^{n} |f_j(z)|^p$ is subharmonic in D
   (b) if there is a $z_0 \in D$ such that $\sum_{j=1}^{n} |f_j(z_0)|^p \geq \sum_{j=1}^{n} |f_j(z)|^p$ for all $z \in D$, then f_j is constant for $j = 1, 2, \cdots, n$.

Hint:
For part (a), discuss separately, centered at 0 then MVP, centered at non-zero then $e^{\ln|f|}$ trick.
For part (b), maximum principle prove the sum is a const, then apply Laplacian on it directly.

S 2017 – 8

8. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on the unit disk $D(0, 1)$ such that

$$F(z) = \sum_{n=1}^{\infty} |f_n(z)|$$

defines a continuous function in $D(0, 1)$ and $F(0) \geq F(z)$ on $D(0, 1)$. Prove f_n are constant for all $n = 1, 2, 3, ...$

Hint:
Totally the same (even more specific) as previous case (S 2014 - 5).

F 2015 – 5

Find a real valued function $u(z)$ that is continuous in the closed disc $\overline{D(0,R)}$ (that is, closed disc centered at 0 of radius $R>0$) and harmonic in $D(0,R)$, and satisfies

$$u(Re^{i\theta}) = \frac{1}{2}(1+\cos^3\theta), \quad \theta \in [0, 2\pi).$$

Hint:
Poisson integral formula for harmonic function.

S 2017 – 4

4. Let $f_n : D(0,1) \to D(0,1) \setminus \{0\}$ be analytic such that $\sum_{n=1}^{\infty} |f_n(0)| < \infty$.
(a) Prove $\sum_{n=1}^{\infty} |f_n(z)|^3$ converges uniformly on $|z| \le \frac{1}{2}$;
(b) Give an example of $\{f_n\}_{n=1}^{\infty}$ satisfying above conditions but $\sum_{n=1}^{\infty} |f_n(z)|^3$ diverges for any $|z| > 1/2$.

Hint:
Nonzero function on $\mathbb{D}$, create $g := -\ln|f|$ which is a nonnegative harmonic function. In this way, we can apply Harnack inequality on g. This process also can be directly achieved through polar expression of f.

S 2019 - 2

2. Let $u : \mathbf{C} \to \mathbf{R}$ be a nonconstant real harmonic function. Show that there exists a sequence of points $\{z_n\} \in \mathbf{C}$ such that $\lim_{n\to\infty} u(z_n) = -\infty$.

Hint:
PBC. Create e^{-f} where $f = u + iv$ to use Liouville.

F 2012 – 9

9. Let u be a real-valued harmonic function in $\mathbf{C}\backslash\{0\}$. Show that then

$$u(z) = c \log |z| + \mathrm{Re}\,(f(z))$$

for some real constant c and a holomorphic function f on $\mathbf{C}\backslash\{0\}$.

Hint:
Before you even start doing this, let me tell you a background story about this problem. When Dr. Ziang Long asked Prof. Anton Gorodetski how to deal with this problem around 2 or 3 years ago (2017/2018), this great Russian-American mathematician thought about it for around a minute and said, "this is a standard fact, but I do not have the answer on the top of my head". Hence obviously the answer of this problem is the fact that, if Gorodetski can't figure out in a minute, then we don't even need to touch it. Just leave it alone, try something else that makes your life easier.

Special Topic: Domain Extension

F 2012 – 2

2. Suppose f is analytic in an annulus $A(0, r, R) = \{z \in \mathbf{C} : r < |z| < R\}$, and there exists a sequence of polynomials p_n converging to f uniformly on compact subsets of $A(0, r, R)$. Show that f is an analytic function on the disc $D(0, R)$.

S 2013 – 6, S 2015 - 8

Suppose f is analytic in an annulus $r < |z| < R$, and there exists a sequence of polynomials p_n converging to f uniformly on any compact subset of the annulus. Show that f is an analytic function on the disc $\{\, |z| < R\}$.

F 2016 – 2

2. Suppose f is analytic in the annulus $A = \{z \in \mathbf{C} : 1/2 < |z| < 2\}$, and there exists a sequence of polynomials p_n converging to f uniformly on the unit circle $|z| = 1$. Show that f can be extended to be an analytic function on the disc $D(0, 2)$.

Method 1: max mod principle + Montel theorem (+identity principle)
Method 2: Cauchy integral formula + uniform convergence theorem
Method 2 is better. If we can show the coefficients of the expansions of those polynomials converges to the coefficients of the Laurent expansion of f, then we done as the Laurent expansion has no principal part so analytic. The key to show that is using integral expression of the coefficient, i.e. using general Cauchy formula to represent derivatives of f in the expressions of coefficients. Then, we have uniformly convergence inside of the integral, so by UCT, we can move limit sign out and done.

F 2014 – 8, F 2016 - 8

8. Let $u(z)$ be harmonic in $D =: D(0, 1) \setminus \{0\}$ such that

$$\lim_{z \to 0} \frac{u(z)}{\log |z|} = 0$$

Prove that u can be extended to be harmonic in $D(0, 1)$.

S 2019 – 8

8. Let u be harmonic in $D(0, 1) \setminus \{0\}$ satisfying

$$\lim_{z \to 0} \frac{u(z)}{\ln |z|} = 0.$$

Prove that u is harmonic on $D(0, 1)$.

Hint:
Very constructive.
Start from the Dirichlet problem, we get the solution, and need to show u agrees with

that solution, by constructing 2 functions.

S 2015 – 7

TRUE or FALSE: There exists a bounded harmonic function on the upper half plane $\mathbb{H}$ that cannot be extended to any larger domain. Explain your answer.

Hint:
$\mathbb{H}$ is the same as $\mathbb{D}$, i.e. conformally equivalent. For $\mathbb{D}$, create one s.t. can't be extended from that point, then use weighted rotation to distribute that unextendible point around $\partial\mathbb{D}$ densely.

Special Topic: Quick Facts

Q: for a harmonic $u(z)$, can we always find its harmonic conjugate?
A: Yes, as long as the domain is simply connected.

Q: what's the situations of $1/z$ and $\sqrt{z}$, say, on the $\mathbb{D}\backslash\{0\}$?
A: $1/z$ is holomorphic on the punctured unit disc, it just has no antiderivative, because $Res_f(\infty) \neq 0$ so FTC doesn't work. $\sqrt{z}$ is not holomorphic on $\mathbb{D}$, as its derivative undefined at 0. $\sqrt{z}$ is not holomorphic on $\mathbb{D}\backslash\{0\}$ as well, it has multiple branches, so the correct domain to make it holomorphic is a Riemann surface not traditionally on $\mathbb{C}$. So $\sqrt{z}$ is only holomorphic on the simply connected sets where it defined.

Q: what's the type of singularity of 0 for $z^{-\frac{1}{2}}$? Removable, pole or essential?

A: this is just an undefined point, it is not a singularity. The definition of singularity

requires function to be holomorphic around it, i.e. analytic in a neighborhood of 0. However, again, because the branches of $\sqrt{z}$, such neighborhood doesn't exist.

Q: can we use uniqueness principle (aka identity principle) on any domain?
A: yes, as long as the domain is connected. Which means, simply connected is not required! Holes are allowed on the domain.

Q: when can we use Harnack inequality or Harnack principle?
A: as long as the function is non-negative harmonic. Subharmonic doesn't work.

Q: when can we use $Res_f(z_0) = \frac{P(z_0)}{Q'(z_0)}$ (P-Q type) to compute residue?

A: $f(z) = \frac{P(z)}{Q(z)}$, where z_0 is a simple zero of $Q(z)$. That's it. No matter $P(z)$ and $Q(z)$ are polynomials or not, $P(z) = 0$ or $\neq 0$, we can always use it as long as z_0 is a simple zero of $Q(z)$.

Q: if ∞ is removable for $f(z)$, is $Res_f(\infty) = 0$ necessarily true?

A: No. Even if $\lim_{z\to\infty} f(z) = 1 < \infty$, $Res_f(\infty) \neq 0$ is still possible.

Q: $\Delta = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$?

A: that's right. Another expression for Laplacian operator applied on some forms of $|f|^p$ where f is holomorphic.

Q: can Poisson integral formula be used anywhere for harmonic function?
A: No. The typical formula is only designed for harmonic function on the unit circle centered at the origin. However, you can equip a moving + scaling conformal map composites in that harmonic function, where you can easily get the modified formula.

Q: how can we understand Schwarz reflexive lemma in general version?
A: start with the classical version: $f(z) = \overline{f(\bar{z})}$ where z belongs to lower half plane. The extended function value at that point is $\overline{f(\bar{z})}$, where $\bar{z}$ represent the point on the upper half plane symmetric to z w.r.t the real line. The extended value of f at z is

the conjugate function value of its symmetric point. So as long as the function takes real function values at some lines (or even curves), then we can use reflexive lemma, where we only need to find the certain symmetric point, take its known function value, do the conjugate, and then all set. More generally, the function values on some lines may not be real, but mapped to some other lines on $\mathbb{C}$ instead. Those cases are little more complicated. We need to equip conformal maps and vice versa to see what's really going on here.

Q: are there any useful equivalent statements for $z \in \mathbb{R}$?
A: $\bar{z} = z$. This is super useful as it can also be applied on $\bar{f}(z) = f(z)$ is f map on real values on some certain lines, especially for Taylor expansion of f.

Q: some easy-to-forget theorems?
A: Hurwitz theorem about nowhere vanishing convergence and zeros, theorem 5.2.2 about local image-preimage and injectivity issues, argument principle about orders of singularity/zero, the second inequality of Schwarz-Pick lemma of function value estimates, legitimation of holomorphic logarithm lemma about exponential expression of nonvanishing function on simply connected domain, Poisson integral formula for harmonic function with the hard-memoizable Poisson kernel, Harnack principle for ascending harmonic functions, Jensen's formula for counting zeros.