UCI Complex Analysis Qualifying Solutions

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July 3, 2020

Acknowledgement
This work cannot be done without great help from
Professor Anton Gorodetski and John Nick Treuer

1 Spring 2020

Problem 1.

Suppose f is an entire holomorphic function and

$$\iint_{\mathbb{R}^2} |f(x+iy)|^2 dxdy < \infty.$$

Prove that $f \equiv 0$.

Proof:

$$\int_{\mathbb{C}} |f(z)|^2 dA(z) = \iint_{\mathbb{R}} |f(x+iy)|^2 dxdy < \infty$$

so say $\exists\ M \leq 0$, *s.t.* $\int_{\mathbb{C}} |f(z)|^2 dA(z) = M < \infty$ bounded.

$\forall z \in \mathbb{C}$, by **Mean Value Principle**[1],

$$f^2(z) = \frac{1}{\pi R^2} \int_{D(z,R)} f^2(w) dA(w)$$

holds for any $R > 0$ as f is entire.

So

$$|f(z)|^2 \leq \frac{1}{\pi R^2} \int_{D(z,R)} |f(w)|^2 dA(w) \leq \frac{1}{\pi R^2} \int_{\mathbb{C}} |f(w)|^2 dA(w) = \frac{M}{\pi R^2}$$

because radius R is arbitrary, so $|f(z)|^2 \leq \frac{M}{\pi R^2} \to 0$ as $R \to \infty$.

Hence $|f(z)|^2 \equiv 0 \Rightarrow |f(z)| \equiv 0 \Rightarrow f(z) \equiv 0\ \forall z \in \mathbb{C}$. $\square$

[1] MVP Ver.2, 2 dimensional case

Problem 2.

Suppose f is a holomorphic function on the disk $D(0, R)$ about the origin with radius $R > 1$, and that $|f(z)| < 1$ for $|z| = 1$. Prove that there exists a unique point z_0 in the disk such that $f(z_0) = z_0$.

Proof:
Let $g(z) := f(z) - z$ and $h(z) := -z$, then for any $z \in \partial D(0, 1)$, i.e. $|z| = 1$,

$$|g(z) - h(z)| = |f(z)| < 1 = |z| = |h(z)|,$$

so by **Rouche Theorem**,

$$\#\mathbb{Z}_{D(0,1)}(g) = \#\mathbb{Z}_{D(0,1)}(h) = 1,$$

i.e. $g = f - z$ has an unique root in the unit disk, which means that unique root is exactly its unique fixed point s.t. $f(z_0) = z_0$. $\square$

Problem 3.

Consider the function $f(z) = z\cos\frac{\pi}{z}$. Find $\int_\gamma f(z)dz$, where the closed curve γ is given by $\gamma(t) = e^{2it} + \frac{1}{2}e^{3it}$, $t \in [0, 2\pi]$.

Sol:
$f(z)$ has one singularity 0. Let's start with computing the index of function f around 0:

$$\begin{aligned} Ind_f(0) &= \frac{1}{2\pi i}\int_\gamma \frac{1}{z}dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{\gamma(t)}\gamma'(t)dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{2ie^{2it} + \frac{3i}{2}e^{3it}}{e^{2it} + \frac{1}{2}e^{3it}}dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{4 + 3e^{it}}{2 + e^{it}}dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{4 + 3e^{it}}{2e^{it} + e^{2it}}ie^{it}dt \\ &= \frac{1}{2\pi i}\int_{|z|=1} \frac{4 + 3z}{2z + z^2}dz \end{aligned}$$

the only singularity of $\frac{4+3z}{2z+z^2}$ in $D(0,1)$ is 0, a simple pole. So

$$Res_{\frac{4+3z}{2z+z^2}}(0) = \frac{4 + 3z}{2 + 2z}\Big|_{z=0} = 2,$$

by **Residue Theorem**,

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{4 + 3z}{2z + z^2}dz = \frac{1}{2\pi i}\cdot 2\pi i \cdot 2 = 2,$$

so $Ind_f(0) = 2$. By expansion:

$$\cos\frac{\pi}{2} = 1 - \frac{1}{2}\cdot(\frac{\pi}{2})^2 + \frac{1}{24}\cdot(\frac{\pi}{z})^4 + ...$$

so

$$f(z) = z\cos\frac{\pi}{2} = z - \frac{\pi^2}{2}\cdot\frac{1}{z} + \frac{\pi^4}{24}\cdot\frac{1}{z^3} + ...$$

by the uniqueness of Laurent expansion,

$$Res_f(0) = a_{-1} = -\frac{\pi^2}{2}$$

so by **Residue Theorem**,

$$\begin{aligned}\int_\gamma f(z)dz &= 2\pi i \cdot Ind_f(0) \cdot Res_f(0) \\ &= 2\pi i \cdot 2 \cdot -\frac{\pi^2}{2} \\ &= -2\pi^3 i\end{aligned}$$

Problem 4.

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant nowhere vanishing entire holomorphic function. Let $U = \{z \in \mathbb{C} : |f(z)| < 1\}$. Show that U is unbounded.

Proof:
Assume U is bounded. Because f is entire and $f(z) \neq 0$ for any $z \in \mathbb{C}$, i.e. nowhere vanishing. So let $g(z) := \frac{1}{f(z)}$ is also an entire holomorphic function, and $U = \{z \in \mathbb{C} : |f(z)| < 1\} = \{z \in \mathbb{C} : |g(z)| > 1\}$ is bounded, which means exist bounded closed set V s.t. $U \subseteq V$. So

$$\max_{z \in V} |g(z)| = \max_{z \in \partial V} |g(z)| = M < \infty$$

for some $M > 0$ by **Maximum Modulus Theorem**.

On the other hand, for any $z \in \mathbb{C} \backslash V$, because $z \notin U$, so $|g(z)| \leq M$ on $\mathbb{C}$ uniformly bounded. By **Liouville Theorem**, g is a constant, so $f = \frac{1}{g}$ is also a constant, contradict to f non-constant! Hence, U should be unbounded. $\square$

Problem 5.

Let f be a function holomorphic in the open unit disc, bounded by 1 in absolute value, and vanishing at the origin. Show that the sum

$$\sum_{n=1}^{\infty} f(z^n)$$

converges uniformly on compact subsets of the unit disc to a holomorphic function.

Proof:
Because $|f(z)| \leq 1$ on $\mathbb{D} = D(0,1)$, so

$$f : \mathbb{D} \to \mathbb{D},\ 0 \mapsto 0,$$

so by **Schwarz Lemma**, $|f(z)| \leq |z|$ on $\mathbb{D}$.

For any compact subset Ω of $\mathbb{D}$, $\exists\ R$ where $0 < R < 1$, s.t. $\Omega \subseteq D(0,R) \subseteq \mathbb{D}$. So on Ω:

$$|\sum_{n=1}^{\infty} f(z^n)| \leq \sum_{n=1}^{\infty} |f(z^n)| \leq \sum_{n=1}^{\infty} |z^n| = \sum_{n=1}^{\infty} |z|^n \leq \sum_{n=1}^{\infty} R^n < \infty$$

as $R < 1$.

Hence $\sum_{n=1}^{\infty} f(z^n)$ converges uniformly and absolutely on any compact subsets of $\mathbb{D}$. $\square$

Additional Notes: *The uniform limit of holomorphic functions is holomorphic. $f(z^n)$ holomorphic, so the partial sum $\sum_{n=1}^{N} f(z^n)$ is also holomorphic, so $\sum_{n=1}^{\infty} f(z^n) = \lim_{N\to\infty} \sum_{n=1}^{N} f(z^n)$ converges to a holomorphic function.*

Problem 6.

Prove or disprove:

a) There exists a continuous real valued non-constant function u on the square $\Pi = \{0 \leq Rez \leq 1,\ 0 \leq Imz \leq 1\}$, harmonic in its interior, such that $u(z) = 0$ for each $z \in [0, 1]$.

b) There exists a continuous complex valued non-constant function f on the square $\Pi = \{0 \leq Rez \leq 1,\ 0 \leq Imz \leq 1\}$, holomorphic in its interior, such that $f(z) = 0$ for each $z \in [0, 1]$.

Sol:
a) TRUE.
Consider $u(z) = u(x + iy) = y$, u is non-constant and $\Delta u \equiv 0$ so harmonic. When $z \in [0, 1] \subseteq \mathbb{R}$ then $u(z) = y = 0$.

b) FALSE.
Assume such holomorphic $f(z)$ exists. Because $f(z) = 0 \in \mathbb{R}$ on $[0, 1]$, by **Schwarz Reflexive Principle**, $f(z)$ can be extended to be holomorphic on $\{0 \leq Rez \leq 1,\ -1 \leq Imz \leq 1\}$ by

$$\tilde{f}(z) := \begin{cases} f(z) & z \in \{0 < Rez < 1,\ 0 < Imz < 1\} \\ 0 & z \in (0, 1) \subseteq \mathbb{R} \\ \overline{f(\bar{z})} & z \in \{0 < Rez < 1,\ -1 < Imz < 0\} \end{cases}$$

However, $\tilde{f}(z) = 0$ on $(0, 1)$ which is an accumulative set, by **Uniqueness Principle**[2], $\tilde{f}(z) \equiv 0 \Rightarrow f(z) \equiv 0$, contradict with $f(z)$ non-constant! $\square$

[2]also known as **Identity Principle**, **Isolated Zeros Theorem**, etc.

Problem 7.

Prove or disprove: there exists a sequence $\{f_n\}$ of holomorphic functions on the unit disc $D(0,1)$ such that $f_n \to \bar{z}^3$ uniformly on compact subsets of $D(0,1)$.

Claim: the statement is FALSE.

Assume such holomorphic sequence exists, while $\partial D(0, \frac{1}{2})$ is a compact subset of $D(0,1)$, so on it:

$$f_n \xrightarrow{uniformly} \bar{z}^3 \;\Rightarrow\; z^2 f_n \xrightarrow{uniformly} z^2\bar{z}^3$$

$$z^2\bar{z}^3 = z^2\bar{z}^2\bar{z} = |z|^2\bar{z} = |z|^2 \cdot \frac{|z|}{z} = |z|^3 \cdot \frac{1}{z} = \frac{1}{8z}$$

On the other hand,

$$\int_{\partial D(0,\frac{1}{2})} z^2 f_n dz = 0, \;\forall n \in \mathbb{N}$$

by **Cauchy Integral Theorem** as $z^2 f_n$ holomorphic. So

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_{\partial D(0,\frac{1}{2})} z^2 f_n dz \\ &= \int_{\partial D(0,\frac{1}{2})} \lim_{n\to\infty} z^2 f_n dz \\ &= \int_{\partial D(0,\frac{1}{2})} \frac{1}{8z} dz \\ &= \frac{\pi i}{4} \neq 0 \end{aligned}$$

contradict!

So such sequence doesn't exist. □

Problem 8.

Fix $0 < \gamma < 1$ and suppose f is a holomorphic function on the unit disc satisfying

$$|f(z) - f(w)| \leq C_1 |z - w|^\gamma$$

for some constant C_1. Show that there exists a constant $C_2 > 0$ such that

$$|f'(z)| \leq C_2 (1 - |z|)^{\gamma - 1}.$$

Proof:
For any fixed $z \in \mathbb{D} = D(0, 1)$, define function of w:

$$g(w) := \frac{f(z) - f(w)}{z - w}$$

Since f is holomorphic on $\mathbb{D}$, g is holomorphic on $D(z, R) \setminus \{z\} \subseteq \mathbb{D}$ for any R where $0 < R \leq 1 - |z|$. Since

$$\lim_{w \to z} g(w) = \lim_{w \to z} \frac{f(z) - f(w)}{z - w} = f'(z) < \infty,$$

by **Riemann Removable Singularity Theorem**, z is removable for g, so g is actually holomorphic on $D(z, R)$ with extended value $g(z) := f'(z)$.

On the other hand, since $|f(z) - f(w)| \leq C_1 |z - w|^\gamma$, so

$$|g(w)| = \left| \frac{f(z) - f(w)}{z - w} \right| \leq C_1 |z - w|^{\gamma - 1}$$

for all $w \in D(z, R)$. So for $w \in \partial D(z, R)$, $|g(w)| \leq C_1 R^{\gamma - 1}$. Now, by **Maximum Modulus Principle**,

$$|f'(z)| = |g(z)| \leq \max_{w \in \partial D(z, R)} |g(w)| \leq C_1 R^{\gamma - 1}.$$

Because g is holomorphic in $D(z, R)$ for any R s.t. $D(z, R) \subseteq \mathbb{D}$, so the inequality above should hold for any satisfied R. Recall that $0 < R \leq 1 - |z|$, so

$$|f'(z)| \leq C_1 (1 - |z|)^{\gamma - 1},$$

hence $C_2 = C_1$ is exactly what we desired. $\square$

The complete file of all Qual solutions can be requested in person only for some policy reasons.