Witelski 1998.

(3.13) steady state : $u_t \equiv 0$.

(a.k.a equilibrium)

$\Rightarrow$ (3.13a) $\left(f(u) - \varepsilon^2 u_{xx}\right)_{xx} = 0$

$\Rightarrow$ (3.14) $f(u) - \varepsilon^2 u_{xx} = ax + b$

by perturbation expansion on u.

the leading order case i.e. $O(1)$ case

$$f(u) = ax + b$$

coupled with (3.13b) $u(0) = 1$, $u(1) = 0$

implicitly, $x(1) = 0$, $x(0) = 1$

(reversely)

$\Rightarrow$ $x = 0$, $u = 1$: $b = f(1)$

$x = 1$, $u = 0$: $a + b = f(0)$

$\Rightarrow a = f(0) - f(1)$

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So back to the leading order case

$$f(u) = ax + b$$

$$= (f(0) - f(1))x + f(1)$$

implicitly, $$x = \frac{f(u) - f(1)}{f(0) - f(1)} \qquad (3.15)$$

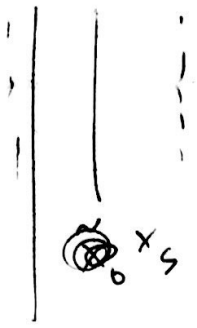
note that the values of a and b are well-determined by leading order equation, this is NOT an approximation!

So we have (3.16):

$$f(u) - \varepsilon^2 u_{xx} = (f(0) - f(1))x + f(1)$$

the multivalued outer solution only agree with $\varepsilon = 0$ case, so we said it cannot satisfy for $\varepsilon > 0$.

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$\tilde{x} = \frac{x - x_s}{\varepsilon}$ ← some position in the shock

$$\frac{d\tilde{x}}{dx} = \frac{1}{\varepsilon}$$

$$x = \varepsilon \tilde{x} + x_s$$

$$\tilde{u}_x = \tilde{u}_{\tilde{x}} \frac{d\tilde{x}}{dx} = \frac{1}{\varepsilon} \tilde{u}_{\tilde{x}}$$

$$u_{xx} = \tilde{u}_{xx} = \frac{d}{dx}(\tilde{u}_x) = \frac{d}{dx}\left(\frac{1}{\varepsilon} \tilde{u}_{\tilde{x}}\right)$$

$$= \frac{1}{\varepsilon} \cdot \frac{d}{dx}(\tilde{u}_{\tilde{x}})$$

$$= \frac{1}{\varepsilon} \tilde{u}_{\tilde{x}\tilde{x}} \cdot \frac{d\tilde{x}}{dx}$$

$$= \frac{1}{\varepsilon^2} \cdot \tilde{u}_{\tilde{x}\tilde{x}}$$

i.e. $u_{xx} = \frac{1}{\varepsilon^2} \cdot \tilde{u}_{\tilde{x}\tilde{x}}$ put into (3.16)

$$f(\tilde{u}) - \tilde{u}_{\tilde{x}\tilde{x}} = (f(0) - f(1))(\varepsilon \tilde{x} + x_s) + f(1)$$

$$= \underbrace{(f(0) - f(1)) x_s + f(1)}_{c} + O(\varepsilon,$$

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$\Rightarrow$ (3.17) $f(\tilde{u}) - \tilde{u}_{\tilde{x}\tilde{x}} = c + O(\varepsilon)$

again, check leading order equation for this inner layer case:

i.e. $O(1)$ case

$$\tilde{u}_{\tilde{x}\tilde{x}} = f(\tilde{u}) - c$$

~~as this is implicit.~~

try: $\dfrac{d(\tilde{u}_{\tilde{x}})}{d\tilde{x}} = f(\tilde{u}) - c$

$\Rightarrow d(\tilde{u}_{\tilde{x}}) = (f(\tilde{u}) - c)\, d\tilde{x}$

integrating both sides from $\tilde{x}_0$ to $\tilde{x}$

Seems not right.

try: verify (3.19)

do derivative on both sides w.r.t $\tilde{u}$

$$\Rightarrow \frac{d\tilde{x}}{d\tilde{u}} = -\frac{1}{\sqrt{2(F(\tilde{u}) - c\tilde{u})}}$$

$$\Rightarrow \frac{d\tilde{u}}{d\tilde{x}} = -\sqrt{2(F(\tilde{u}) - c\tilde{u})}$$

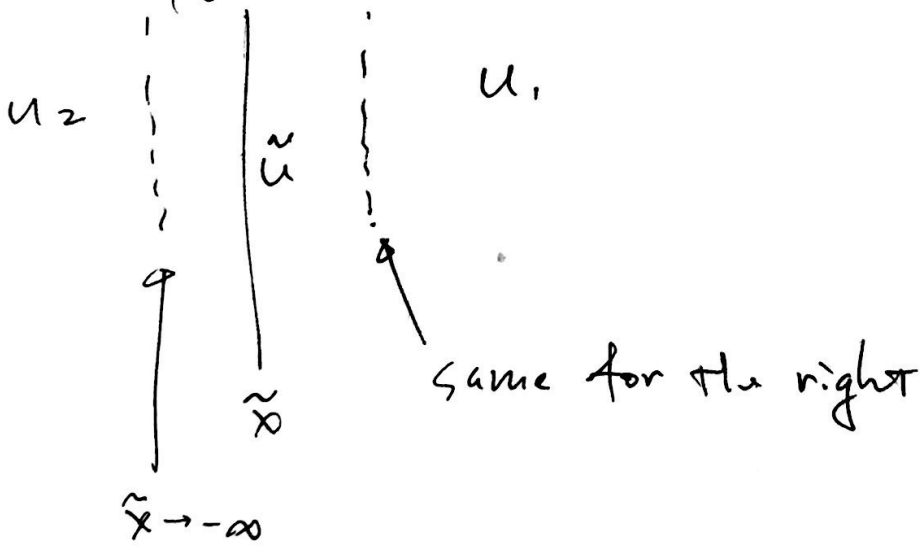
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$$\Rightarrow \frac{d^2\tilde{u}}{d\tilde{x}^2} = -\sqrt{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{F(\tilde{u})-c\tilde{u}}}\cdot\left(f(\tilde{u})\cdot\tilde{u}_{\tilde{x}}-c\tilde{u}_{\tilde{x}}\right)$$

$$u_{xx} = f(u) - c \qquad \text{why (3.19)?}$$

---

shock. (inner layer)



$\tilde{x} \to -\infty$

as x far away from x_s
on the left i.e. $x - x_s = O(1)$

so $\tilde{x} = \frac{x-x_s}{\varepsilon} \to -\infty$

and $\tilde{u} \to u_2$ left jump value.

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from (3.16) get rid of terms with ε
the outer solution should satisfy

$$f(u) = (f(0) - f(1))x + f(1)$$

Now, x is very close to the shock
(actually, $|\tilde{x}| \to \infty$ means x just out of the shock, so in the limit sense,

left side $f(u_1) = (f(0) - f(1))x_s + f(1) = c$
right side $f(u_2) = (f(0) - f(1))x_s + f(1) = c$ } (3.21)

in (3.19), if we required denominator vanish at u_1 and u_2 (boundary of the shock)

$$F(u_j) - c u_j = 0 \iff \int_0^{u_j} (f(u) - c)\,du = 0$$

(3.20) $\Uparrow$ $\int_0^{u_j} f(u)\,du$ $\quad$ $\int_0^{u_j} c\,du$

because $F(0) = 0$?.

$\Downarrow$

(3.22) equal area rule.

necessary cond. ?

Def: u_s is the one s.t. $x(u_s) = x_s$
where x_s in (3.18) has

$$x_s = \frac{c - f(1)}{f(0) - f(1)} = x(u_s)$$

$\sim$ (3.15), $\boxed{c = f(u_s)}$

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the dynamic problem: introduce time variable.

consider $u_t = (f(u) - \varepsilon^2 u_{xx})_{xx}$

from the leading order equation of the inner problem, we can derive the same equal-area rule. (3.22)

Now study outer sol of (3.24)

$u_t = f(u)_{xx} \iff u_t = (D(u) u_x)_x$

diffusive flux $q := -D(u) u_x$

$\Rightarrow$ (3.32) $u_t + q_x = 0$

similarity solutions $u(x,t) = U(z)$ with $z = \frac{x}{\sqrt{t}}$

$$u_t = U'(z) \frac{\partial z}{\partial t} = -\frac{1}{2} x \cdot t^{-\frac{3}{2}} U'(z) = -\frac{1}{2} \cdot \frac{1}{t} \cdot z \cdot U'(z)$$

$$z = x \cdot t^{-\frac{1}{2}} \quad \frac{\partial z}{\partial t} = -\frac{1}{2} x \cdot t^{-\frac{3}{2}}$$

$$\frac{\partial z}{\partial x} = t^{-\frac{1}{2}}$$

derivative w.r.t z

$$u_x = U'(z) \frac{\partial z}{\partial x} = t^{-\frac{1}{2}} U'(z)$$

$$D(u) u_x = D(U) \cdot U'(z) \cdot t^{-\frac{1}{2}}$$

$$(D(u) u_x)_x = (D(U) \cdot U'(z))' \cdot \frac{\partial z}{\partial x} \cdot t^{-\frac{1}{2}}$$

$$= (D(U) \cdot U'(z))' \cdot \underbrace{t^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}}}_{\frac{1}{t}}$$

$\Rightarrow$ (3.34) $\frac{1}{2} z U'(z) = -(D(U) U'(z))'$

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rewrite the diffusive flux

$$q = -D(u)u_x = \phi^{-\frac{1}{2}} Q(z) \quad (3.35)\ (3.36)$$

$$D(U)U'(z)$$

$$\Rightarrow (3.37)\ \frac{1}{2} z U'(z) = Q'(z)$$

Now we ~~turn~~ turn our original nonlinear diffusion PDE into our current similarity ODE

Now, turn $U = U(z) \longrightarrow z = z(U)$

rewrite (3.37)

$$\Rightarrow \frac{1}{2} z \cdot \frac{dU}{dz} = \frac{dQ}{dz} \Rightarrow \frac{1}{2} z = \frac{dQ}{dz} \cdot \frac{dz}{dU} = \frac{dQ}{dU}$$

$$\Rightarrow \frac{1}{2} z(U) = \frac{dQ}{dU} \quad (3.38)$$

inverse flux $\bar{D}(U) = \frac{1}{Q(U)} = -\frac{1}{D(U)} \cdot \frac{dz}{dU} = -\frac{z'}{D(U)}$.

$$\Rightarrow (3.40)\ \frac{1}{2} z = \left(\frac{1}{J}\right)' = -\left(\frac{z'}{D}\right)' \quad \text{derivative w.r.t } U$$

$$\Rightarrow \frac{1}{2} z = -\left(\frac{z'' D(U) - z' D'(U)}{D^2(U)}\right)$$

$$\Rightarrow \frac{1}{2} z \cdot (-D^2(U)) = D(U) z'' + D'(U) z'$$

$$z'^2 \quad (3.41)$$

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derivative w.r.t U

retry from (3.40)

$$\frac{1}{2}\chi = \left(\frac{1}{J}\right)' = -\frac{1}{J^2}\cdot J'$$

$$= -\frac{1}{\frac{z'^2}{D^2}}\cdot\left(-\frac{z''D - z'D'}{D^2}\right)$$

$$= \frac{D^2}{z'^2}\cdot\frac{z''D - z'D'}{D^2}$$

$$= \frac{z''D - z'D'}{z'^2}$$

$$\Rightarrow \quad \frac{1}{2}\chi z'^2 = D(U)z'' - D'(U)z' \qquad (3.41)$$

$\Updownarrow$ if $D(u) > 0$

(3.34)

$U(x_*^+) = 0$

$\Downarrow$

Leading edge of the region of support

how to determine?

global conservation of mass.

from (3.37), integrating from 0 to x_*

$$(3.43)\quad \frac{1}{2}\int_0^{x_*} \chi U'(z)\,dz = Q(x_*) - Q(0)$$

Recall Q is the similarity flux

and the flux vanishes at the leading edge x_*

means $Q(x_*) = 0$

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How about $Q(0)$?

Note here $Q(0)$ means $Q|_{x=0}$ on the boundary according to the boundary condition

$u(0,t) \equiv 1 \quad \forall\, t>0$

so $Q|_{x=0} = Q|_{U=1} = -D(1)\left.\frac{dU}{dx}\right|_{x=0} = -D(1)\left.\frac{1}{\frac{dx}{dU}}\right|_{U=1}$

↑ by $Q(x) = -D(U)U'(x)$,

so $Q|_{x=0} = -\frac{D(1)}{x'(1)}$ ← here, variable is U instead of x.

so (3.43) RHS $= 0-\left(-\frac{D(1)}{x'(1)}\right) = \frac{D(1)}{x'(1)}$

$$\text{LHS} = \frac{1}{2}\int_0^{x_*} x\cdot\frac{dU}{dx}\,dx$$

$$= \frac{1}{2}\int_1^0 x(U)\,dU$$ ← because after changing of var. upper bd $x = x_* \Leftrightarrow U = 0$, lower bd $x = 0 \Leftrightarrow U = 1$

$$= -\frac{1}{2}\int_0^1 x(U)\,dU$$

$$\text{LHS} = \text{RHS} \Leftrightarrow -\frac{1}{2}x'(1)\int_0^1 x(U)\,dU = D(1) \quad (3.44)$$

↑ these results hold for general $D(U)$ but if $D(0)>0$ i.e. $D|_{U=0}>0$, then the similarity solutions do not have compact support and $x_* \to \infty$

why? [7][21]

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instead of integrating from 0 to x_+,
we integrate from x_+ to some indefinite x.
then we'll get generalized (3.44)

(3.45) $D(U) = -\frac{1}{2}x'(U)\int_0^U x(U)\,du$

Now let's turn to $D(U) < 0$ case.

[2] [16] got Fig. 7.

the motion of shock $x_s = s(t)$,

$$\frac{ds}{dt} = \frac{[\xi]}{[u]} = -\frac{[f(u)_x]}{[u]} \quad \text{as } \xi = -f(u)_x$$

$s(t) = x_s\sqrt{t}$ → do derivative w.r.t t on both sides.

$$\frac{ds}{dt} = \frac{1}{2}x_s \cdot \frac{1}{\sqrt{t}}$$

$$\Rightarrow \frac{1}{2}x_s = \frac{[Q]}{[U]} = -\frac{[D(U)U']}{[U]} \quad (3.50)$$

$$\frac{[q]}{[u]} = \frac{[\frac{1}{\sqrt{t}}Q]}{[U]} = \frac{[Q]}{[U]}\cdot\frac{1}{\sqrt{t}}$$

(3.55)

recall (3.34) reduced ODE: $\frac{1}{2}zU'(z) = -(D(U)U'(z))'$

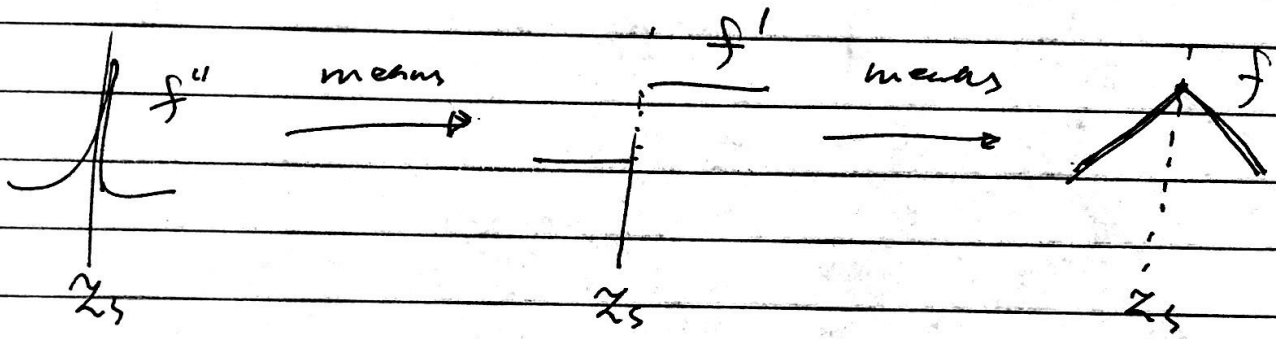
$$= -f''(U) \quad (3.51)$$

here, $'$ is w.r.t z.

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$$(3.51)\ LHS = \frac{1}{2}x\frac{d}{dx}\left(U_2(x)H(x_s-x) + U_1(x)H(x-x_s)\right)$$
$$= \frac{1}{2}x\left(U_2'(x)H(x_s-x) - U_2(x)\delta(x_s-x)\right.$$
$$\left. + U_1'(x)H(x-x_s) + U_1(x)\delta(x-x_s)\right)$$

so f'' has singularity like a delta function at x_s.



delta type step type C^0 type.

$\Rightarrow [f(U)] = 0$

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